

Deriving the properties of space time using the non-compressible solutions of the Navier-Stokes Equations

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Recent observations of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) has confirmed one of the last outstanding predictions in general relativity and in the process opened up a new frontier in astronomy and astrophysics. Additionally the observation of gravitational waves has also given us the data needed to deduce the physical properties of space time. Bredberg et al have shown in their 2011 paper titled From Navier-Stokes to Einstein, that for every solution of the incompressible Navier-Stokes equation in $p + 1$ dimensions, there is a uniquely associated dual” solution of the vacuum Einstein equations in $p + 2$ dimensions. The author shows that the physical properties of space time can be deduced using the recent measurements from the Laser Interferometer Gravitational-Wave Observatory and solutions from the incompressible Navier-Stokes equation.

I. INTRODUCTION

Albert Einstein first published his theory of general relativity in 1915, a year later he would predict the existence of gravitational waves [1]. He based this prediction on his observation that the linearized weak field equations had wave solutions [2][3], these transverse waves would be comprised of spatial strain and travel at the speed of light and would be generated by the time variations of the mass quadrupole moment of the source. Einstein also observed that the amplitude of these waves would be incredible small and thus he never expected to observe them [1]. That same year Karl Schwarzschild would publish his solutions to the Einstein field equations that would predict the existence of black holes and in 1963 he would generalize these solutions to rotating black holes[4][5].

In 2016 the Laser Interferometer Gravitational-Wave Observatory(LIGO) announced they had detected a gravitational wave on September 14, 2015 at 09:50:45 UTC, The signals frequency ranged from 35 to 250 Hz with a peak gravitational-wave strain of $1 * 10^{-21}$ [1]. This wave matched the waveform predicted by general relativity for the in spiral and merger of a pair of black holes and the ringdown of the resulting single black hole.

The detection of such a wave is important on its own because the measurement of the waveform and amplitude of the gravitational waves from a black hole merger (such as the one observed by LIGO) event makes possible accurate determination of its distance. The accumulation of black hole merger data from cosmologically distant events may help to create more precise models of the history of the expansion of the universe and the nature of the dark energy that influences it. However the direct detection of gravitational waves also allows use to determine the physical properties of space, by using solutions common to both the incompressible Navier-Stokes equation and the Einstein field equations of general relativity

to calculate a hypothetical viscosity of space time and thus deduce the structure.

Before moving forward with the analysis, it must be understood that the Viscosity of space time that the author is intending to calculate, is an analogues property rather than a literal one. Meaning that rather than trying to estimate the thickness of a fluid like space time, we intend to show that by calculating the viscosity term in a Navier-Stokes like equation, it is possible to calculate a Shear modulus that describes the deformation of space time, thus allowing for the analysis of space time at its most basic level. Before this can be done though, it is necessary to understand how the solutions of the Navier-Stokes equation relate to the solutions of the Einstein field equation and how the data from the 2016 observations at the Laser Interferometer Gravitational-Wave Observatory (LIGO) can be used to calculate derive the hypothetical properties of space time.

II. THE NAVIER-STOKES EQUATION AND EINSTEIN FIELD EQUATION

Both the Einstein field equation $G_{\mu\nu} = 0$ and the Navier-Stokes equation $\dot{v}_i - \eta \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$ have long been incredibly powerful tools for understanding the world both terms of mathematics and physics. The Einstein equation universally governs the long-distance behavior of gravitational systems, while the incompressible Navier-Stokes equation universally governs the hydrodynamic limit of essentially any fluid. Both equations have an intricate non-linear structure that allow them to be applied to wide verity of systems and thus offer us an insight into a wide range of physical phenomena. Before these two equation can be used to probe the structure of space time though it necessary to understand that for that for every solution of the incompressible Navier-Stokes equation in $p+ 1$ dimensions, there is a uniquely associated dual” solution of the vacuum Einstein equations in $p + 2$ dimensions[6]. Those interested in a more rigorous demonstration of this assertion should read From Navier-Stokes to Einstein by Dr. Irene Bred-

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berg et al[6].

III. THE 2016 OBSERVATION OF GRAVITATIONAL WAVES

On September 14th 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) directly observed gravitational waves from the inward spiral and merger of a pair of black holes of around 36 and 29 solar masses [1]. Up until then existence of gravitational waves had only been inferred indirectly, through their effect on the timing of pulsars in binary star systems [8]. The observation of these waves gives us the information we need to use the Navier-Stokes equation to calculate the hypothetical viscosity of space time, primarily it allows us to calculate the frequency and amplitude of the gravitational wave. This calculation was possible prior to the detection of Gravitational waves, however with the direct observation of the waves, we can now use data that is physically meaningful.

The two black holes involved in the merger had masses of approximately 36(+5, -4) and 29(+ or - 4) solar masses[1]. The recorded power of the gravitational wave peaked at 3.6×10^9 watts and Across the 0.2-second duration of the detectable signal, the relative tangential (orbiting) velocity of the black holes increased from 30

We begin with the equation for the space time curvature expressed with respect to a covariant derivative ∇ , in the form of the Einstein tensor $G_{\mu\nu}$. We can then relate the curvature to the stressenergy tensor $T_{\mu\nu}$, with the equation $G_{\mu\nu} = \frac{8\pi G_n}{c^4} T_{\mu\nu}$ [9]. Where G_n is Newton's gravitational constant, and c is the speed of light, for the purpose of this paper we will use geometrized units so that $G_n = 1 = c$ in order to simplify our calculations. We can now rewrite the Einstein's equations as wave equations and define our flat space, our flat space time will be given as

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

(it should be noted that in the next section η [9] will also be used to calculate our hypothetical viscosity of space time.) This flat-space metric has no physical significance; it is a purely mathematical device necessary for the analysis.

Now, we can also think of the physical metric $g_{\mu\nu}$ as a matrix and find its determinant $\det g$. Finally we define our radiation field in terms of our flat space time $g_{\mu\nu}$ and find the determinant $\det g$, this will give us $\bar{h}^{\alpha\beta} \equiv \eta^{\alpha\beta} - g^{\alpha\beta} \sqrt{|\det g|}$. Next we define our coordinates in such a way that this quantity satisfies the de Donder gauge condition (conditions on the coordinates): $\nabla_\beta \bar{h}^{\alpha\beta} = 0$. Where ∇ represents the flat-space derivative operator. These equations say that the divergence of

the field is zero. The linear Einstein equations can now be written as $\square \bar{h}^{\alpha\beta} = 16\pi \tau^{\alpha\beta}$ [10] where $\square = -\partial_t^2 + \Delta$ is the flat-space d'Alembertian operator, and $\tau^{\alpha\beta}$ represents the stressenergy tensor plus quadratic terms involving $\bar{h}^{\alpha\beta}$ [10]. This is just a wave equation for the field with a source, despite the fact that the source involves terms quadratic in the field itself. That is, it can be shown that solutions to this equation are waves traveling with velocity 1 in these coordinates.

In order to obtain a numerical result from this equation though it will need to be linearized. We assume that space is nearly flat, so the metric is nearly equal to the $\eta^{\alpha\beta}$ tensor. This means that we can neglect terms quadratic in $\bar{h}^{\alpha\beta}$, which means that the $\tau^{\alpha\beta}$ field reduces to the usual stressenergy tensor $T^{\alpha\beta}$ and Einstein's equations become $\square \bar{h}^{\alpha\beta} = 16\pi T^{\alpha\beta}$. However since we are interested in the field far from a source, we can treat the source as a point and everywhere else, the stressenergy tensor would be zero, so our equation becomes $\square \bar{h}^{\alpha\beta} = 0$. Now, this is just the usual homogeneous wave equation for each component of $\bar{h}^{\alpha\beta}$. For a wave moving away from a point source, the radiated part (meaning the part that dies off as $1/r$ far from the source) can always be written in the form $\frac{f(t-r, \theta, \phi)}{r}$, where f is just a generic function. It can be shown that it is always possible to make the field traceless[11].

Now, if we further assume that the source is positioned at $r=0$, the general solution to the wave equation in spherical coordinates is

$$\bar{h}^{\alpha\beta} = \frac{1}{r} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_+(t-r, \theta, \phi) & h_\times(t-r, \theta, \phi) \\ 0 & 0 & h_\times(t-r, \theta, \phi) & h_+(t-r, \theta, \phi) \end{bmatrix}$$

[11]. Thus we have our polarization equations which can now be rewritten in Newtonian terms as $h_+ = \frac{-1}{R} \frac{G^2}{c^4} \frac{4m_1 m_2}{r}$ [9] and to calculate the amplitude of the gravitational wave detected by the Laser Interferometer Gravitational-Wave Observatory we simply substitute the measured values and get an upper value of $25956/R$ or a lower value of $19890/R$, where R is the distance from the center of the system. The collision was detected approximately 1.3 billion Light years away from earth [1], and thus the spatial distortion caused by the gravitational wave was between 2.1×10^{-21} meters and 1.6×10^{-21} meters. This is the amplitude of the gravitational wave detected by the Laser Interferometer Gravitational-Wave Observatory and with the amplitude of the wave, we can now calculate the hypothetical properties of space time by using the Navier-Stokes equation.

IV. THE THEORETICAL PROPERTIES OF SPACE TIME AS FLUID

We have now established that for that for every solution of the incompressible Navier-Stokes equation in $p+1$ dimensions, there is a uniquely associated dual" solution of the vacuum Einstein equations in $p+2$ dimensions [6] and calculated the amplitude of the wave 2016 gravitational wave. We also know that the gravitational wave detected by The Laser Interferometer Gravitational-Wave Observatory has velocity of c [1]. With this information we can now use the Navier-Stokes equation to derive an equation for the velocity of wave in an incompressible medium and calculate the Shear modulus. We begin by considering a wave as disturbance of a fluid-like medium[12][13][14].

Next we write out the equations for the conservation of mass and momentum as $\frac{\partial \rho u_i}{\partial x_i} + \frac{\partial \rho}{\partial t} = 0$ and $\frac{\partial \rho u_i}{\partial x_j} u_j + \frac{\partial \rho u_i}{\partial t} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0$ [12][13][14] where ρ is the mass density (for our calculations this will be the smallest possible deformation in space time) u is the particle velocity, P is our fluid pressure. It should again be noted that when we refer to things like mass, particle and pressures, we are speaking metaphorically about analogous variables in continuum mechanics. The physical meaning of each of the analogous variables will be covered in the next section. We also have our three spatial coordinates are x_i ($i = 1, 2, 3$) for the domain ω . Particle velocities u are for each direction x_i . Using the Navier-Stokes equation we can find the equation for incompressible fluid flow $\rho \frac{du}{dt} = f_b - \nabla P$ and the body forces are negligible giving us $f_b = 0$ [12].

The force $f(t)$ acts as a source function upon the domain ω . Any force acting within the domain causes pressure and density changes, and the fluid nature of the medium will create an equilibrium restoring force[15]. Next we consider small perturbations Δ in the density ρ , the particle velocity u , and the pressure P from the initial rest conditions which are labeled with subscript 0. $u_t = u_0 + \Delta u$, $\rho_t = \rho_0 + \Delta \rho$, $P_t = P_0 + \Delta P$ [15]. The initial particle velocity of our location in space is assumed to be 0 because the domain ω is assumed to be at rest relative to the particles around it. The density perturbation is based on the acoustic approximation[15], in this approximation A fluid has a pressure which is a function of density, temperature, and gravitational forces.

We shall assume that the gravity forces are relatively constant over the domain ω and do not exert any differential force on the fluid. We will neglect the effects of temperature as the changes are very small. Hence, we will assume that only the density is important and that the stress within the fluid is related to the strain as a function of density. Next we simply apply the Kronecker delta to our stress matrix

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and use Eulers equation to get $\rho \frac{du}{dt} = -\nabla P_t$ and since the pressure is assumed to be constant the equation becomes $\rho \frac{du}{dt} = -\nabla \Delta P$ [16]. Now the initial medium is at rest and has no convective acceleration, which permits changing the form of the derivatives and this gives us $\rho \frac{\partial u}{\partial t} = -\nabla \Delta P$. Next u is the gradient of ϕ , and we can also see that the product of $\Delta \rho$ and the gradient of ϕ will be small thus $\nabla \phi = u$ and $\rho_0 \frac{\partial \nabla \phi}{\partial t} = -\nabla \Delta P$ [12].

Next we assume the derivatives of time and space can be exchanged which will give us $\rho_0 \frac{\partial \phi}{\partial t} = -\nabla \Delta P$ and this equation can be simplified to $\rho_0 \frac{\partial \phi}{\partial t} = -\Delta P$. Next since we are measuring space time in terms of its deformation caused by waves, we use the shear modulus instead of the Young's modulus that would normally be used for calculating the velocity of a wave. To calculate the wave velocity, we take the incompressible fluid equation and apply mass conservation to the first derivative of the fluid equation and then solve for velocity which gives us $u = \sqrt{\frac{G}{\rho_0}}$ (where G is the shear modulus). Next we calculate the shear modulus for space time $G = \frac{Fl}{A\Delta x}$ where F is the force of the two observed black holes colliding, l is the initial length of the smallest area effected by the force, A is the area over which the force acts (this would be the circular distance between earth and the colliding black holes) and Δx would be the spatial displacement caused by the gravitational wave (the amplitude of the gravitational wave we calculated earlier).

We now know all of the quantities for our velocity equation except for the initial length of the smallest area affected by the wave. By substituting the known quantities into the wave velocity equation and solving for l , we find that the smallest area affected by the observed gravitational wave was 1.4×10^{-21} meters. We can now take this information and substitute our results into the equation for dynamic viscosity $F = \mu A \frac{u}{y}$ and solve for μ to get our hypothetical viscosity of space time of 6.2616×10^{-39} kg/ms. This for all intents and purposes gives us a substance with a viscosity that is functionally 0, and thus space time behaves as a perfect fluid.

V. IMPLICATIONS

Before continuing to the consequences of space time behavior as a perfect/super fluid, let us first summarize our argument. We have already established that for every solution of the incompressible Navier-Stokes equation in $p+1$ dimensions, there is a uniquely associated dual" solution of the vacuum Einstein equations in $p+2$ dimensions. This means that any solution derived from Navier-Stokes equation in $p+1$ dimensions there is a dual solution of the vacuum Einstein equations in $p+2$ dimensions. This establishes that our work in one set of equations is valid in the other assuming that both treat their respective mediums as incompressible and that a constant number of dimensions is used in each.

We then take the polarization equation for a system

of orbiting bodies and calculate the amplitude of the gravitational wave caused by this system using the values measured by the Laser Interferometer Gravitational-Wave Observatory in 2015 and then using a shear module and the equation for the velocity of a transvers wave in a medium derived from the Navier-Stokes equation and then solve for the initial length of the space prior to its distortion. We then take the force of the event that caused the spatial distortion, the initial size of the undistorted space, the area affected by the gravitational wave and then solve for the viscosity of space. The resulting viscosity tells us about how malleable space time is and as a result imposes constraints on the size of any extra dimensions space. In this context "viscosity" simply means the deformability of space time, our particles are simply quantized units of space, and the "pressure" is a uniform force present through out all of space time.

According to general relativity, the conventional gravitational wave is the small fluctuation of curved space time which has been separated from its source and propagates independently. In Superfluid vacuum theory, a subset of Grand Unified Field Theories where the fundamental physical vacuum (non-removable background) space time is viewed as a Superfluid and the curved space time we see in general relativity is the small collective excitation of a superfluid background[17]. Meaning that much like water, space time is composed of smaller parts. However the properties of these parts will vary widely depending on what formulation of Superfluid vacuum theory is being used and if Superfluid vacuum theory is being incorporated into another theory such as m-theory. However when treating space time as a super fluid and assuming that general relativity's interpretation of a gravitational wave is accurate, results in the following conclusions about the graviton. Primarily, the Graviton would be the "small fluctuation of the small fluctuation" meaning it would be the fundamental unit of space. This may not appear to be a physically robust concept as it is akin to trying to define a phonon in terms of smaller fluctuations inside it. However it may be possible to better ascertain the nature of such a strange phenomenon if we make the following assumptions about space times struc-

ture as a fluid.

First we assume that there are more than 3+1 dimension of space and that any extra dimensions of space can exist both at the quantum level and on the macroscopic level. Next we assume there are differences in the malleability of these dimensions of space time, meaning that higher dimensions are less malleable then the dimensions we experience every day. With these assumption in place, in theory we should be able to calculate the size of these extra dimension using the EinsteinSmoluchowski relation[18]. However this is where we encounter our first problem with trying to deduce the size of these extra dimensions via the characteristics of gravitational wave. If we simply take the general case of the EinsteinSmoluchowski relation and calculate the mobility μ of our particles of space for the velocity of the expanding universe at a given distance (calculated using Hubble's law) over the force that caused the gravitational wave in conjunction with the ambient temperature of the vacuum, we can then solve for the diffusion constant, the above values give space time a diffusion constant of $8.309710^{-57}m^2/s$.

We then take the StokesEinstein equation[19] and set it equal to the diffusion constant that we just calculated and then substitute our viscosity for η and then simply solve for r . If we do this then we get a radius of $3.79910^{70}m$ to put this in perspective consider that the radius of the observable universe is $4.4 * 10^{26}m$. This would seem to indicate that any extra dimension of space would have to be large extra dimensions akin to those in the RandallSundrum model[20]. However we initially intended to calculate the size of compactified dimensions such as those in m-theory. However in order to do this we need to calculate the drift velocity of the individual units that in theory would comprise space time, however at this level, determining the exact the velocity of any individual unit becomes impossible due to the laws of quantum physics. Thus while our calculations on the macroscopic level introduce the intriguing possibility of large extra dimensions of space, in order to use the data measured by the Laser Interferometer Gravitational-Wave Observatory to deduce the nature of compactified extra dimensions of space, a more complete description of space time at the quantum level is required.

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- [1] BP Abbott, Richard Abbott, TD Abbott, MR Abernathy, Fausto Acernese, Kendall Ackley, Carl Adams, Thomas Adams, Paolo Addesso, RX Adhikari, et al. Observation of gravitational waves from a binary black hole merger. *Physical review letters*, 116(6):061102, 2016.
 - [2] A Einstein. Näherungsweise integration der feldgleichungen der gravitation, 22 jun 1916. 1916.
 - [3] Albert Einstein. Über gravitationswellen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)*, Seite 154-167., 1918.
 - [4] Karl Schwarzschild. On the gravitational field of a point mass according to einsteins theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math.*, 1916.
 - [5] Roy P Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Physical review letters*, 11(5):237, 1963.
 - [6] Irene Bredberg, Cynthia Keeler, Vyacheslav Lysov, and Andrew Strominger. From navier-stokes to einstein. *Journal of High Energy Physics*, 2012(7):1–18, 2012.
 - [7] Irene Bredberg, Cynthia Keeler, Vyacheslav Lysov, and Andrew Strominger. Wilsonian approach to fluid/gravity duality. *Journal of High Energy Physics*, 2011(3):1–29, 2011.
 - [8] Joel M Weisberg and Joseph H Taylor. Relativistic bi-

- nary pulsar b1913+ 16: thirty years of observations and analysis. *arXiv preprint astro-ph/0407149*, 2004.
- [9] Bernard Schutz. *A first course in general relativity*. Cambridge university press, 2009.
 - [10] Kip S. Thorne. Multipole expansions of gravitational radiation. *Rev. Mod. Phys.*, 52:299–339, Apr 1980.
 - [11] If Tredcr et al. Cw misner, ks thorne, ja wheeler: Gravitation. wh freeman and company limited, reading (england) 1973xxvi+ 1279 seiten, preis£ 19.20 (clothbound);£ 8.60 (paperbound). *Astronomische Nachrichten*, 296(1):45–46, 1975.
 - [12] Philip McCord Morse and Hermann Feshbach. *Methods of theoretical physics*. 1953.
 - [13] Weng Cho Chew. *Waves and fields in inhomogeneous media*, volume 522. IEEE press New York, 1995.
 - [14] TJ Chung. *Finite element analysis in fluid dynamics*. NASA STI/Recon Technical Report A, 78:44102, 1978.
 - [15] LD Landau and E Mo Lifshits. Mechanics of continuous media. *Gostekhizdat, Moscow*, 81, 1954.
 - [16] John Artley. *Fields and configurations*. Holt, Rinehart and Winston, 1965.
 - [17] Konstantin G Zloshchastiev. Spontaneous symmetry breaking and mass generation as built-in phenomena in logarithmic nonlinear quantum theory. *arXiv preprint arXiv:0912.4139*, 2009.
 - [18] MA Islam. Einstein–smoluchowski diffusion equation: a discussion. *Physica Scripta*, 70(2-3):120, 2004.
 - [19] Thomas G Mason. Estimating the viscoelastic moduli of complex fluids using the generalized stokes–einstein equation. *Rheologica Acta*, 39(4):371–378, 2000.
 - [20] H Davoudiasl, JL Hewett, and TG Rizzo. Bulk gauge fields in the randall–sundrum model. *Physics Letters B*, 473(1):43–49, 2000.